Entropy production rate of turbulence(The 9th Symposium on Non-Equilibrium Statisitical Physics)

Shibata, Hiroshi

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Hiroshi Shibata

Department of General Education, Faculty of Engineering,
Sojo University, Kumamoto 860-0082, Japan

Abstract

Entropy production rate has a close relationship with mean Lyapunov exponent. The statistics of mean Lyapunov exponent is studied for turbulence described by the complex Ginzburg-Landau equation. The result shows that the probability distribution function of finite time average for mean Lyapunov exponent is the same form as for Kuramoto-Sivashinsky equation.

Tel +81-96-326-3111
Fax +81-96-326-3000
Electronic address:shibata@ed.sojo-u.ac.jp

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1.Introduction

The Lyapunov exponent is a key word to understand the nonlinear dynamical systems. In recent years, the relationship of the Lyapunov exponent with physical observables has been studied extensively[1-5]. One of them is the relationship with the entropy production rate.

On the other hand, the characterization of turbulence is one of the most important issues in physics[6-12]. The characterization of turbulence through statistics is not clear even now. We take up here the statistics of mean Lyapunov exponent for turbulence. The mean Lyapunov exponent has a close relationship with thermodynamical entropy production rate. We study the characteristics of turbulence by the statistics of mean Lyapunov exponent for turbulence.

In this Introduction, we define the mean Lyapunov exponent and discuss the relationship of mean Lyapunov exponent with thermodynamical entropy production rate.

First we define the mean Lyapunov exponent[13, 14]. Let us assume a set of variables at the discrete time n as

$$\vec{X}_n = (x_n(1), x_n(2), \cdots, x_n(N-1), x_n(N))^T, \tag{1}$$

where the system size is N and T means the transpose. $\vec{X}_n$ is assumed to be given by $\vec{X}_{n-1}$ through smooth function. Then we add a small perturbation $\delta\vec{X}_n$ to $\vec{X}_n$ and study the response $\delta\vec{X}_{n+1}$ of $\vec{X}_{n+1}$. The mean Lyapunov exponent is defined as

$$\lambda_n \equiv \frac{1}{N} \ln \left| \frac{\delta\vec{X}_{n+1}}{\delta\vec{X}_n} \right|, \tag{2}$$

where

$$\frac{\delta\vec{X}_{n+1}}{\delta\vec{X}_n} = \begin{pmatrix} \frac{\partial x_{n+1}(1)}{\partial x_n(1)} & \frac{\partial x_{n+1}(1)}{\partial x_n(2)} & \frac{\partial x_{n+1}(1)}{\partial x_n(3)} & \cdots & \frac{\partial x_{n+1}(1)}{\partial x_n(N)} \\ \frac{\partial x_{n+1}(2)}{\partial x_n(1)} & \frac{\partial x_{n+1}(2)}{\partial x_n(2)} & \frac{\partial x_{n+1}(2)}{\partial x_n(3)} & \cdots & \frac{\partial x_{n+1}(2)}{\partial x_n(N)} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \frac{\partial x_{n+1}(N)}{\partial x_n(1)} & \frac{\partial x_{n+1}(N)}{\partial x_n(2)} & \frac{\partial x_{n+1}(N)}{\partial x_n(3)} & \cdots & \frac{\partial x_{n+1}(N)}{\partial x_n(N)} \end{pmatrix} \tag{3}$$

and $|\cdot|$ means the determinant.

Second we discuss the relationship of the mean Lyapunov exponent with thermodynamical entropy production rate. Let us assume local equilibrium for the system under study. Thermodynamical entropy is written down as

$$S = k_B \ln W(\{\vec{x}^*\}, t), \tag{4}$$

where W is the number of microscopic states and $\{\vec{x}^*\}$ are the dynamical variables $\{\vec{q}, \vec{p}\}$. k_B is the Boltzmann constant. From Eq.(4), we obtain

$$\frac{dS}{dt} = \frac{k_B}{W}\left[\frac{\partial W}{\partial t} + \sum_l \dot{x}^*(l) \frac{\partial W}{\partial x^*(l)}\right]. \tag{5}$$

On the other hand, the existence of measure demands the equation of continuity

$$\frac{\partial W}{\partial t} = -\sum_l \frac{\partial}{\partial x^*(l)} (W \dot{x}^*(l)). \tag{6}$$

Substituting Eq.(6) to Eq.(5) we obtain

$$\frac{dS}{dt} = -k_B \sum_l \frac{\partial \dot{x}^*(l)}{\partial x^*(l)} = -k_B \sum_l \lambda_l^*. \tag{7}$$

Eq.(7) is transformed to the time discrete version

$$S_{n+1} - S_n = -k_B C(1) N \lambda_n, \tag{8}$$

where λ_n is the mean Lyapunov exponent in Eq.(2) and $C(1)$ is a constant.

2.Mean Lyapunov exponent of turbulence

We take up the complex Ginzburg-Landau equation that describes both the phase turbulence and the defect turbulence. The complex Ginzburg-Landau equation is written down as

$$\partial_t A = A + (1 + iC_1)\partial_x^2 A - (1 - iC_3)|A|^2 A, \tag{9}$$

where ∂_t and ∂_x are partial differential operators with respect to time t and space x, respectively. A is a complex variable. When we write A_R for the real part of A and A_I for the imaginary part of A, A_R and A_i are subjected to the following equations

$$\partial_t A_R = A_R + \partial_x^2 A_R - C_1 \partial_x^2 A_I - (A_R^2 + A_I^2)A_R - C_3(A_R^2 + A_I^2)A_I \tag{10a}$$

and

$$\partial_t A_I = A_I + C_1 \partial_x^2 A_R + \partial_x^2 A_I + C_3(A_R^2 + A_I^2)A_R - (A_R^2 + A_I^2)A_I. \tag{10b}$$

Solving Eq.(10) with finite difference method we obtain the mean Lyapunov exponent through Eq.(2). The Jacobi matrix of the discretized version for Eq.(10) is

$$J_n = \begin{pmatrix}
\frac{\partial \psi_1^{n+1}}{\partial \psi_1^n} & \frac{\partial \psi_1^{n+1}}{\partial \psi_2^n} & \cdots & \frac{\partial \psi_1^{n+1}}{\partial \psi_N^n} & \frac{\partial \psi_1^{n+1}}{\partial \phi_1^n} & \frac{\partial \psi_1^{n+1}}{\partial \phi_2^n} & \cdots & \frac{\partial \psi_1^{n+1}}{\partial \phi_N^n} \\
\frac{\partial \psi_2^{n+1}}{\partial \psi_1^n} & \frac{\partial \psi_2^{n+1}}{\partial \psi_2^n} & \cdots & \frac{\partial \psi_2^{n+1}}{\partial \psi_N^n} & \frac{\partial \psi_2^{n+1}}{\partial \phi_1^n} & \frac{\partial \psi_2^{n+1}}{\partial \phi_2^n} & \cdots & \frac{\partial \psi_2^{n+1}}{\partial \phi_N^n} \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
\frac{\partial \psi_N^{n+1}}{\partial \psi_1^n} & \frac{\partial \psi_N^{n+1}}{\partial \psi_2^n} & \cdots & \frac{\partial \psi_N^{n+1}}{\partial \psi_N^n} & \frac{\partial \psi_N^{n+1}}{\partial \phi_1^n} & \frac{\partial \psi_N^{n+1}}{\partial \phi_2^n} & \cdots & \frac{\partial \psi_N^{n+1}}{\partial \phi_N^n} \\
\frac{\partial \phi_1^{n+1}}{\partial \psi_1^n} & \frac{\partial \phi_1^{n+1}}{\partial \psi_2^n} & \cdots & \frac{\partial \phi_1^{n+1}}{\partial \psi_N^n} & \frac{\partial \phi_1^{n+1}}{\partial \phi_1^n} & \frac{\partial \phi_1^{n+1}}{\partial \phi_2^n} & \cdots & \frac{\partial \phi_1^{n+1}}{\partial \phi_N^n} \\
\frac{\partial \phi_2^{n+1}}{\partial \psi_1^n} & \frac{\partial \phi_2^{n+1}}{\partial \psi_2^n} & \cdots & \frac{\partial \phi_2^{n+1}}{\partial \psi_N^n} & \frac{\partial \phi_2^{n+1}}{\partial \phi_1^n} & \frac{\partial \phi_2^{n+1}}{\partial \phi_2^n} & \cdots & \frac{\partial \phi_2^{n+1}}{\partial \phi_N^n} \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
\frac{\partial \phi_{N-1}^{n+1}}{\partial \psi_1^n} & \frac{\partial \phi_{N-1}^{n+1}}{\partial \psi_2^n} & \cdots & \frac{\partial \phi_{N-1}^{n+1}}{\partial \psi_N^n} & \frac{\partial \phi_{N-1}^{n+1}}{\partial \phi_1^n} & \frac{\partial \phi_{N-1}^{n+1}}{\partial \phi_2^n} & \cdots & \frac{\partial \phi_{N-1}^{n+1}}{\partial \phi_N^n} \\
\frac{\partial \phi_N^{n+1}}{\partial \psi_1^n} & \frac{\partial \phi_N^{n+1}}{\partial \psi_2^n} & \cdots & \frac{\partial \phi_N^{n+1}}{\partial \psi_N^n} & \frac{\partial \phi_N^{n+1}}{\partial \phi_1^n} & \frac{\partial \phi_N^{n+1}}{\partial \phi_2^n} & \cdots & \frac{\partial \phi_N^{n+1}}{\partial \phi_N^n}
\end{pmatrix}. \tag{11}$$

In Eq.(11), the sets $\{\psi_i^n; i = 1, 2, \cdots, N\}$ and $\{\phi_i^n; i = 1, 2, \cdots, N\}$ are discretized variables for $A_R(x, t)$ and $A_I(x, t)$, respectively. The superscript means the discrete time and the subscript means the discrete space. We set periodic boundary conditions

$$\psi_{N+i}^n = \psi_i^n, \quad \phi_{N+i}^n = \phi_i^n, \quad (i = 1, 2, \cdots, N) \tag{12}$$

and initial condition

$$\psi_i^0 = 1.0 \sin(\frac{2\pi}{N} i), \quad \phi_i^0 = 0.0. \quad (i = 1, 2, \cdots, N) \tag{13}$$

In addition, we set the system size length $L = 80\pi$, the system size $N = \frac{L}{\Delta x} = 256$, the time step width $\Delta t = 0.0001$, $C_1 = 3.0$, and $C_3 = 1.5$. In this parameter regime, Eq.(10) describes the defect turbulence. The mean Lyapunov exponent defined by Eq.(2) is rewritten as

$$\lambda_n = \frac{1}{N \times \Delta t} \ln \left| J_n \right|, \tag{14}$$

where $n = \frac{t}{\Delta t}$. The second-order Adams-Bashforth is used in order to solve Eq.(10). At the same time, the second-order central difference of space is adopted. The time series of the mean Lyapunov exponent is shown in Fig.1.

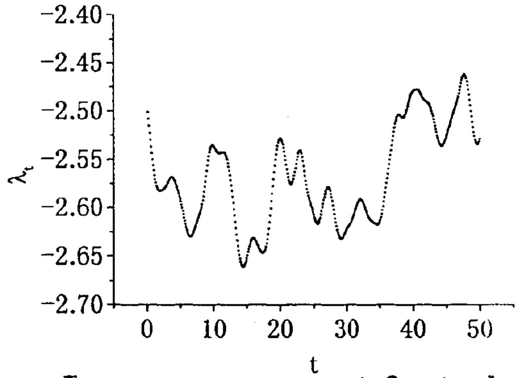

Fig.1

3.Statistics of mean Lyapunov exponent for turbulence

We study the statistics of the mean Lyapunov exponent for the complex Ginzburg-Landau equation. First we define the local Lyapunov exponent as

$$\Lambda_n \equiv \frac{1}{n} \sum_{j=0}^{n-1} \lambda_j. \tag{15}$$

We introduce here the probability distribution function of Λ_n as

$$P(\Lambda; n) = < \delta(\Lambda - \Lambda_n) > . \tag{16}$$

$P(\Lambda; n)$ is the probability that Λ takes the value Λ_n. $P(\Lambda; n)$s are shown in Fig.2 for $n = 2 \times 10^4$(a), 4×10^4(b), and 8×10^4(c).

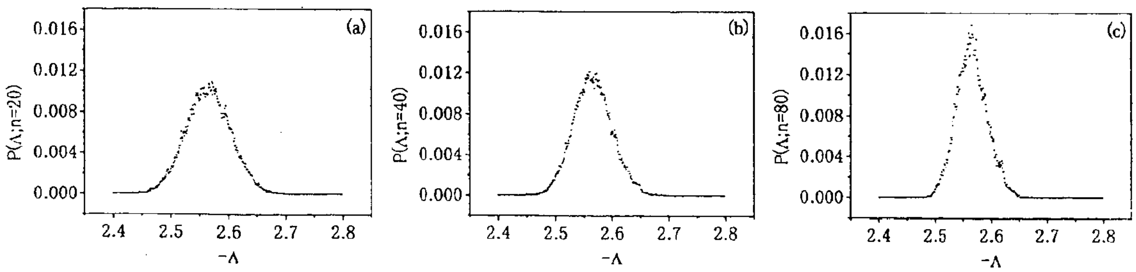

Fig.2

$P(\Lambda; n)$s in Fig.2 collapse a parabola by choosing appropriate value of B as shown in Fig.3.

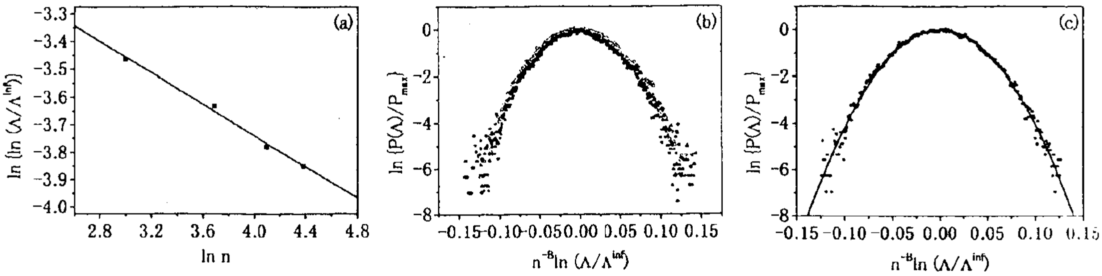

Fig.3

The result is described as follows. There is a relation between

$$y = \ln\{P(\Lambda; n)/P_{max}\} \tag{17}$$

and

$$x = n^{-B}\ln(\Lambda/\Lambda^{\infty}) \tag{18}$$

as

$$y = -bx^2, \tag{19}$$

where P_{max} is the maximal value of $P(\Lambda; n)$ and Λ^{∞} is the value of Λ that gives P_{max}. The values of b and B in this case are 411 and -0.28, respectively. The result is summed up as

$$P(\Lambda; n) = P_{max} \cdot (\Lambda/\Lambda^{\infty})^{-bn^{-2B}\ln(\Lambda/\Lambda^{\infty})}. \tag{20}$$

4.Concluding remarks

The fluctuation of entropy production rate for turbulence can be written as

$$P(\Lambda; n) = P_{max} \cdot (\Lambda/\Lambda^{\infty})^{-bn^{-2B}\ln(\Lambda/\Lambda^{\infty})}.$$

Acknowledgements

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